Macroeconomic Fluctuations and Countercyclical Income Risk*

Alberto Polo†

This version: September 2018
First version: May 2016

Abstract

What are the quantitative implications of countercyclical labor earnings risk? This paper investigates the welfare effects of eliminating business cycles when households face cyclical changes in the skewness of the labor earnings distribution as estimated by Guvenen, Ozkan and Song (2014). Using a heterogenous agent, general equilibrium model with aggregate shocks I find that the average welfare effect can be as large as 9% of lifetime consumption. The welfare gain comes entirely from removing cyclical changes in the distribution of persistent idiosyncratic shocks. At the individual level, the welfare gain is increasing in earnings and decreasing in wealth. Low-earnings, low-wealth households however have little to lose from countercyclical risk and prefer the economy with aggregate fluctuations.

Key words: Countercyclical Risk, Heterogeneous Agents, Welfare Effects of Business Cycles

JEL Codes: E21 E32

*I am grateful to comments by Jaroslav Borovička, Virgiliu Midrigan, José Montiel Olea, Gianluca Violante and participants in the NYU Third Year Paper Workshop. I would also like to thank Spencer Lyon and Bálint Szőke for many useful discussion. All errors are my responsibility only.

†Bank of England. Email: alberto.polo@bankofengland.co.uk

1 Introduction

In recent years large, confidential datasets on labor earnings records have been used by economists to study the distribution of idiosyncratic earnings shocks. Of particular interest is the evidence developed by Guvenen, Ozkan and Song (2014). They document that recessions are not associated with higher variance of the earnings growth distribution - the traditional way of representing an increase in risk. On the contrary, recessions appear to be different from expansions due to a bigger chance of downward movements in individual earnings without an increase in the probability of upward movements. Only in this sense risk is countercyclical. This evolution of risk over the business cycle is captured by cyclical skewness of the earnings growth distribution: skewness tends to increase in expansions and decrease in recessions.

Despite the intuitive conclusions drawn by Guvenen et al. (2014), the importance of the extent of cyclicality in labor earnings growth that they document is yet to be assessed. In principle, with enough wealth, individuals could insure their consumption from the consequences of this cyclicality in the distribution they face. If this were the case, then one could think that excluding such feature from models could be done at an acceptable cost. Moreover, before introducing microfounded mechanisms that can generate the kind and extent of cyclicality in the earnings distribution which is observed in the data, it is crucial to have an idea of the size of the impact that such mechanisms can have.

There are several directions that can be pursued in order to investigate the implications of Guvenen et al. (2014)'s results. This paper focuses on welfare effects of eliminating business cycles. Specifically, it answers the question: what are the welfare effects of eliminating business cycles when households are subject to persistent idiosyncratic shocks with cyclical skewness? The question is answered within a Krusell and Smith (1998) type of model by computing the proportional increase in lifetime consumption that makes a household indifferent between living in the economy with cycles and going through a transition to an economy without cycles. I follow Storesletten, Telmer and

Yaron (2001) by including the impact of the transition period in the comparison, rather than just comparing steady states.

I obtain that the average welfare effect of eliminating business cycles in this model economy is approximately 9% of lifetime consumption. All welfare gains come from eliminating fluctuations in the earnings distribution. Quantitative results are almost unchanged whether it is assumed that household's preferences have constant relative risk aversion or are of the Epstein-Zin type. Allowing unsecured borrowing also does not have a significant impact on the average welfare effect.

Looking at welfare effects for different households, I find that households with low wealth and earnings prefer the economy with aggregate fluctuations to the transition to an economy without business cycles. This happens because, for a household who is close to the lower bound on earnings and wealth, recessions do not have much worse consequences than if business cycles had been removed. However, expansions come with a higher probability of upward movements, and this is what the household likes in that region of the state space.

This option-value effect becomes less important as wealth and earnings increase. Given a low value of earnings, welfare effects increase and turn positive with higher wealth as there is now room for a differential impact of the cyclical process in bad times on household's wealth and eventually consumption. Given a low value of wealth, welfare effects are increasing in earnings as the higher probability of downward movements in earnings during recessions starts to bite. For middle to high values of earnings, welfare effects are decreasing in wealth, while remaining positive. This is due to larger self-insurance capacity afforded by higher initial wealth, so that cyclicality in the idiosyncratic distribution becomes relatively less important for household's consumption.

As usual in this literature, welfare effects quantified through the model have to be considered as an upper bound on welfare effects, because households may well have other sources of self-insurance against idiosyncratic shocks that are not captured in the model, such as income pooling among household members, more borrowing opportunities and default.

A popular strand of literature going back to Lucas (1987) has studied the welfare effects of eliminating business cycles. Storesletten et al. (2001), Krebs (2003, 2007) and Berger, Dew-Becker, Milbradt, Schmidt and Takahashi (2016) in particular have considered such effects in presence of countercyclical changes in idiosyncratic uninsurable risk. However, to my knowledge, there has been no account of the welfare effects of eliminating business cycles in a heterogeneous agents, general equilibrium model with households who are subject to cyclical changes in skewness.

In order to obtain closed-form solutions, Krebs (2003, 2007) and Berger et al. (2015) have considered specific cases where idiosyncratic shocks have permanent effects. Under their assumptions a no-trade equilibrium exists, thus the wealth distribution is degenerate at zero. Welfare costs arise if the cost of job loss is countercyclical (Krebs 2007, using time-separable preferences) and if the probability of job loss is countercyclical (Berger et al. 2015, using Epstein-Zin preferences). Storesletten et al. (2001) is close to this paper as they study welfare effects of eliminating business cycles in a heterogeneous agents, general equilibrium model with aggregate shocks and persistent idiosyncratic shocks. However they focus on countercyclical heteroskedasticity which is not supported by the recent empirical studies.

This paper is also related to the recent literature using mixture-of-Normals processes to model idiosyncratic shocks with non-standard higher moments. In addition to Guvenen et al. (2014), this literature includes Guvenen, Karahan, Ozkan and Song (2015), Civale, Diez-Catalan and Fazilet (2015) and McKay (2016). I build on Civale et al. (2015) and use their method to derive closed-form expressions for higher moments of sums of AR(1) processes with mixture-of-Normals innovations. I share the interest of McKay (2016) in studying the implications of time-varying idiosyncratic risk. However, he focuses on aggregate consumption and permanent - not persistent - idiosyncratic shocks, while considering a much more complex structure for aggregate shocks. This paper instead looks at the implications in terms of welfare effects and only allows for two levels of the single exogenous aggregate state.

The paper is organized as follows. Section 2 describes how the yearly idiosyncratic process estimated by Guvenen et al. (2014) is transformed into a quarterly process, and how the process without business cycles is parameterized. Section 3 deals with the model environment and parameter values adopted. Section 4 explains how the experiment of eliminating business cycles is performed, and quantifies its welfare effects. Section 5 concludes.

2 Idiosyncratic Process

2.1 Cyclical Process

I start with a description of the earnings process estimated in Guvenen et al. (2014) before turning to its approximation.

Guvenen et al. (2014) choose a ”persistent-plus-transitory” specification for log-earnings where errors are distributed according to a mixture of Normals, hence the distribution is allowed to deviate form zero skewness and excess kurtosis. Let s_t^a denote the aggregate state in year t, being it an Expansion (E) or Recession (R). Let the superscript/subscript a denote that the process has annual frequency. The process for yearly log-earnings y_t^a is

$$y_t^a(s_t^a) = z_t^a(s_t^a) + \epsilon_t^a$$

where $z_t^a(s_t^a)$ follows an AR(1) process with mixture-of-Normals innovations $\eta_t^a(s_t^a)$:

$$z_t^a(s_t^a) = \rho_a z_{t-1}^a + \eta_t^a(s_t^a)$$

and ϵ_t^a is the transitory component, being *iid* and independent of all other random variables:

$$\epsilon_t^a \sim \mathrm{N}(0, \sigma_{\epsilon,a})$$

The mixture-of-Normals innovations take the form

$$\eta_t^a(s_t^a) = \xi_t^a \eta_{1,t}^a(s_t^a) + (1 - \xi_t^a)\eta_{2,t}^a(s_t^a)$$

where each component $\eta_{j,t}^a(s_t^a)$ has constant variance between expansion and recession, with the only aggregate-state dependent parameter being the mean:

$$\eta_{j,t}^a(s_t^a) \sim \mathrm{N}(\mu_{j,a}(s_t^a), \sigma_{j,a}) \text{ for } j = 1, 2$$

Finally, mixing is achieved through

$$\xi_t^a \sim \text{Bernoulli}(q_a)$$

Conditioning on the aggregate state the mixture-of-Normals innovations are independent. However, unconditionally, serial correlation in the aggregate state translates into dependence across innovations.

Guvenen et al. (2014) combine a persistent and a transitory component as their estimation strategy is based on minimizing the distance between empirical and model-implied moments of the level y_t^a, first difference $\Delta_1 y_t^a$ (yearly earnings growth) and fifth difference $\Delta_5 y_t^a$ (five-year earnings growth). Fifth difference moments are informative about the persistence parameter ρ_a. High values of persistence however may conflict with level and first difference moments, thus the transitory component is used to dampen short-run serial correlation.

The parameter values estimated by Guvenen et al. (2014) are reported in the following table.

Table 1: Yearly Process Parameters, Guvenen et al. (2014)

ρ_a	0.979	$\mu_{1,a}(E)$	0.119
q_a	0.490	$\mu_{2,a}(E)$	-0.026
$\sigma_{1,a}$	0.325	$\mu_{1,a}(R)$	-0.102
$\sigma_{2,a}$	0.001	$\mu_{2,a}(R)$	0.094
$\sigma_{\epsilon,a}$	0.186		

Table 2 reports central moments of the log-earnings process y_t^a, its first-difference and fifth-difference transformations, conditional on expansion or recession. Moments are computed using the closed-form formulas derived by Civale et al. (2015).

Table 2: Moments of the Earnings Process Estimated in Guvenen et al. (2014)

	Expansion			Recession		
	y_t^a	$\Delta_1 y_t^a$	$\Delta_5 y_t^a$	y_t^a	$\Delta_1 y_t^a$	$\Delta_5 y_t^a$
Mean	2.1452	0.0000	0.0000	-0.0971	0.0000	0.0000
Variance	1.4064	0.1268	0.3454	1.5110	0.1312	0.3665
Skewness	0.1117	0.2546	0.2494	-0.1358	-0.3273	-0.3088
Kurtosis	3.0520	3.5209	3.2984	3.0447	3.4823	3.2627

The table highlights that the key difference between expansion and recession concerns skewness: it is positive conditional on expansion and negative conditional on recession[1].

Since this paper focuses on business cycles, and business cycles are thought of as a quarterly phenomenon, the first step is translating the yearly earnings process into a quarterly process. To this purpose I adopt the same "persistent-plus-transitory" specification also for the quarterly process, then add up quarterly random variables 4 by 4 to obtain a process at annual frequency, and finally solve for parameters of the quarterly process that minimize the distance between the moments of the annualized quarterly process and those reported in Table 2. I derive closed-form expressions for moments of the annualized quarterly process which are similar to those in Civale et al. (2015). Quarterly parameterization is performed through distance minimization because I use more moment targets (20) than parameters (18). Section A in the Appendix contains the closed-form expressions I derive as well as information on accuracy of the distance minimization.

The resulting parameters of the quarterly process are listed in Table 3. These are the parameter values which will be used for the idiosyncratic cyclical process in the model.

[1]Mean log-earnings also change significantly between expansion and recession, but such difference vanishes after normalization.

Table 3: Quarterly Process Parameters

ρ	0.9935	$\mu_1(E)$	0.0230
q	0.2278	$\mu_2(E)$	-0.0023
σ_1	0.0674	$\mu_1(R)$	-0.0266
σ_2	0.0000	$\mu_2(R)$	0.0075
σ_ϵ	0.0842		

Notice that the quarterly persistence parameter ρ increases in magnitude, as the quarterly process has to match moments of 5-year differences. Interestingly, the second Normal in the quarterly mixture innovation becomes degenerate. This is a feature inherited from the yearly process, and it means that, in any period, with probability q the innovation is drawn from a Normal and with probability $1-q$ the innovation is a known value $\mu_2(s_t)$. Hence, if a law of large numbers holds, in any period log-earnings of a fraction $1-q$ of households shift by $\mu_2(s_t)$.

Following the economics literature on estimating income processes, I make the assumption that the labor market is competitive, thus observed differences in individual earnings come from differences in individual productivity. Accordingly, thereafter I refer to the exogenous idiosyncratic variable as "efficiency units". I also normalize mean efficiency units to 1, so that no exogenous aggregate fluctuation is caused by changes in the distribution of efficiency units between expansion and recession. This means that I will be able to evaluate purely the welfare effects of removing cyclical skewness, without other exogenous aggregate implications.

Therfore, let $y_t(s_t)$ be quarterly log efficiency units and $C_{\tilde{y}|s_t}(x)$ be the cumulant generating function of the non-normalized quarterly process $\tilde{y}_t(s_t)$, conditional on aggregate state s_t. The parameter values for both processes are listed in Table 3, the only difference between them being that $y_t(s_t)$ has been normalized[2]. Then the idiosyncratic process used in the model is

[2] I impose the normalization $\mathbb{E}[e^{y_t(s_t)}|s_t = E] = \mathbb{E}[e^{y_t(s_t)}|s_t = R] = 1$. Given $\mathbb{E}[e^{\tilde{y}_t(s_t)}|s_t] = e^{C_{\tilde{y}|s_t}(1)}$, the normalization is $\mathbb{E}[e^{\tilde{y}_t(s_t)-C_{\tilde{y}|s_t}(1)}|s_t] = 1$, or $y_t(s_t) = \tilde{y}_t(s_t) - C_{\tilde{y}|s_t}(1)$. Since $\tilde{y}_t(s_t)$ is not Normally distributed, $C_{\tilde{y}|s_t}(1)$ is an infinite sum. However the summands decay quite fast and I can approximate it to the tenth term, obtaining $C_{\tilde{y}|E}(1) = 0.5846$ and $C_{\tilde{y}|R}(1) = 0.0246$.

$$y_t(s_t) = z_t(s_t) + \epsilon_t$$

$$z_t(s_t) = -(1-\rho)C_{\tilde{y}|s_t}(1) + \rho z_{t-1} + \eta_t(s_t)$$

where

$$\epsilon_t \sim \mathrm{N}(0, \sigma_\epsilon)$$

$$\eta_t(s_t) = \xi_t \eta_{1,t}(s_t) + (1-\xi_t)\mu_2(s_t)$$

$$\eta_{1,t}(s_t) \sim \mathrm{N}(\mu_1(s_t), \sigma_1)$$

$$\xi_t \sim \mathrm{Bernoulli}(q)$$

and its conditional central moments are presented in the following table.

Table 4: Moments of the Quarterly Process for Log Efficiency Units

	Expansion			Recession		
	y_t^a	$\Delta_1 y_t^a$	$\Delta_5 y_t^a$	y_t^a	$\Delta_1 y_t^a$	$\Delta_5 y_t^a$
Mean	-0.0483	0.0000	0.0000	-0.0488	0.0000	0.0000
Variance	0.0958	0.0153	0.0198	0.1025	0.0154	0.0203
Skewness	0.1086	0.0329	0.1071	-0.1329	-0.0441	-0.1405
Kurtosis	3.0533	3.0536	3.1515	3.0513	3.0583	3.1599

Notice that, following the normalization, the only remaining substantial difference between expansion and recession concerns skewness. The mean is still somewhat different because normalization concerns the exponential transformation of the process.

The following two plots show the stationary densities of log-efficiency units y_t and efficiency units e^{y_t}, conditional on expansion and recession. Both highlight the extent to which recessions entail a higher probability of low realizations of earnings. The right tail of the density barely changes between expansions and recessions.

Figure 1: Conditional Stationary Density of Log Efficiency Units y_t

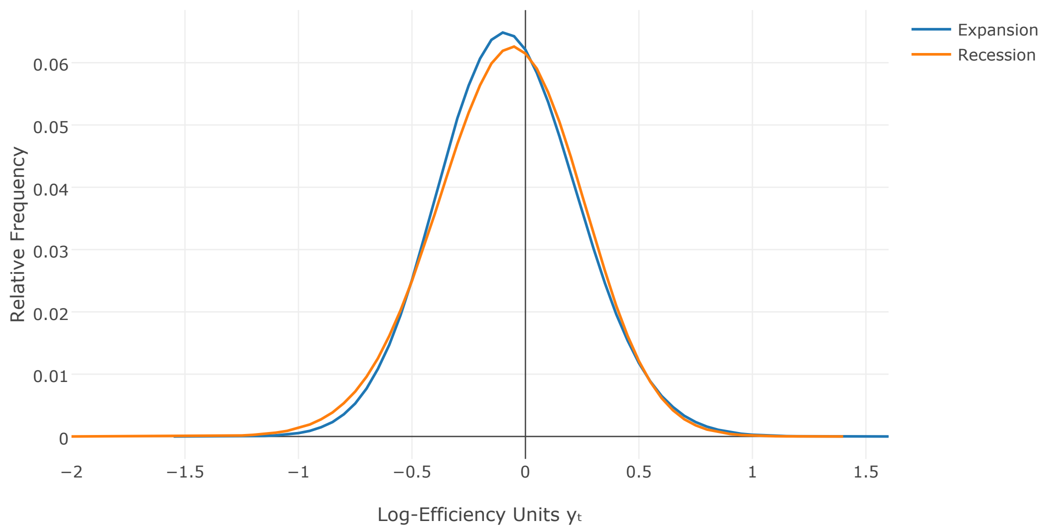
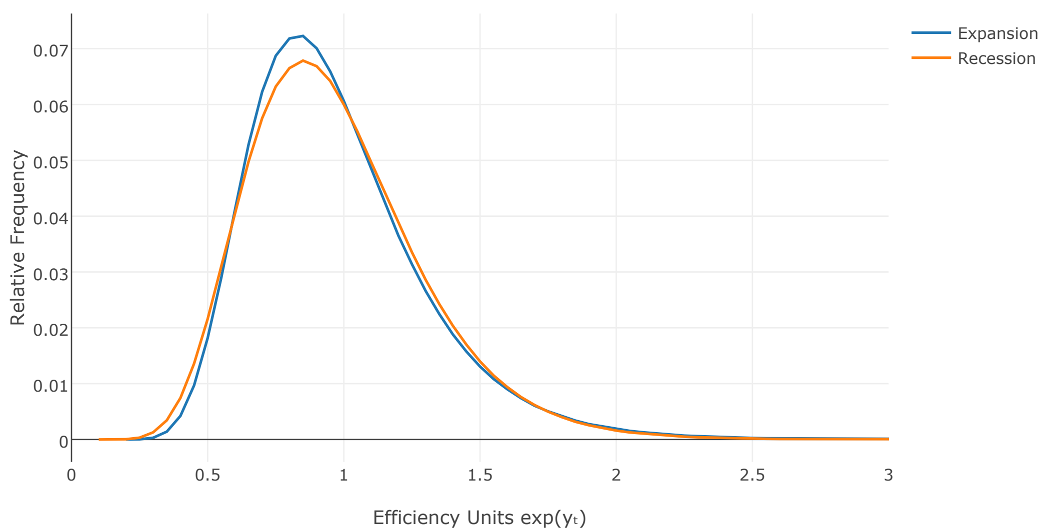
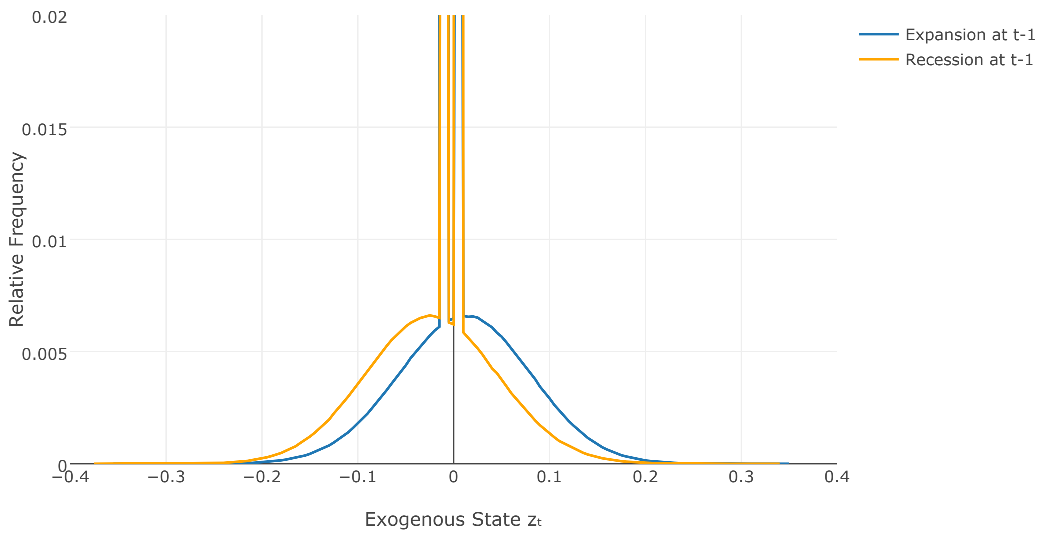


Figure 2: Conditional Stationary Density of Efficiency Units e^{y_t}



In order to better understand the implications of the process that will be used in the model, Figure 3 shows the transition density of the AR(1) component of the process, z_t. The transition density is conditional on the aggregate state in period $t-1$, and is drawn for $z_{t-1} = 0$. However, conditional on the aggregate state, the transition density at different values of z_{t-1} looks similar except for being shifted.

Figure 3: Transition Density of Persistent Component z_t: $p(z_t|s_{t-1}, z_{t-1} = 0)$



The vertical lines correspond to cases when the innovation to z_t is drawn from the second component of the mixture - the deterministic component. These draws are then mixed with the Normal component, which itself will change depending on s_t being an expansion or recession[3]. The main insight of this figure is that, at the level of the persistent component of the idiosyncratic process, expansions or recessions make a significant difference, as recessions come with a higher probability of lower realizations, and a lower probability of high realizations. Notice that the conditional mean is the same regardless of $t-1$ being an expansion or recession. The difference in distributions is purely a matter of skewness.

[3]The transition density depends on the process for the aggregate state, and specifically on the transition probabilities between s_{t-1} and s_t. As described in the following subsection, I assume a Markov chain process for s_t with transition probabilities $P(E|E) = P(R|R) = 0.875$. This is the process used, among others, by Krusell and Smith (1998).

2.2 Acyclical Process

In order to investigate the welfare effects of eliminating business cycles with state-dependent idiosyncratic risk I have to define a process for efficiency units which is independent of expansions and recessions. Ideally, the "Integration Principle" developed by Krusell, Mukoyama, Sahin and Smith (2009) would be used in order to purge the effects of aggregate shocks. However, such method turns out to be of difficult application in the context of a continuous idiosyncratic process as the one at hand.

In eliminating cycles from the idiosyncratic process I follow an approach which is similar to that described in the previous subsection. The cyclical process y_t is itself a mixture between two processes, one for expansion periods and one for recessions. Accordingly, the unconditional distribution of the process is a weighted average between the stationary distributions of the expansion and recession processes, with weights equal to the unconditional probabilities of expansion and recession. I then also compute unconditional moments of the first difference $\Delta_1 y_t$ of the process, taking care of the fact that $(s_t, s_{t-1}) \in \{(E, E), (E, R), (R, E), (R, R)\}$. Then I look for parameter values of a "persistent-plus-transitory" process y_t^{ss} with the same specification used so far - but independent of the aggregate state - such that the unconditional moments of y_t^{ss} and of its first difference $\Delta_1 y_t^{ss}$ match the unconditional moments of y_t and $\Delta_1 y_t$.

In order to determine unconditional moments of the cyclical process y_t I have to define the process for the aggregate state s_t. I assume it follows a Markov chain with the same transition matrix used by Krusell and Smith (1998). Thus $P(s_t = E|s_{t-1} = E) = P(s_t = R|s_{t-1} = R) = 0.875$, implying an average duration of expansions and recessions of 8 quarters.

Appendix B reports additional information on the parameterization of the acyclical process. Notice that the parameters of the acyclical process are exactly identified (7 parameters and 7 moments). Moreover, despite not targeting the unconditional moments of the fifth difference, the unconditional stationary density of the fifth difference is also well matched.

The following tables list parameter values of the acyclical process y_t^{ss} and its unconditional moments, respectively. Comparing to the parameter values of the cyclical process listed in Table 3, the persistence parameters ρ and ρ_{ss}, and the standard deviations of the transitory component σ_ϵ and $\sigma_{\epsilon,ss}$ have the same values. As expected, the parameter values that change are those of the mixture-of-Normals innovations, which are also the components of the process which are aggregate-state dependent[4].

Table 5: Parameters of the Acyclical Process y_t^{ss}

ρ_{ss}	0.9935	$\mu_{1,ss}$	0.0006
q_{ss}	0.7641	$\mu_{2,ss}$	-0.0034
$\sigma_{1,ss}$	0.0004		
$\sigma_{2,ss}$	0.0708		
$\sigma_{\epsilon,ss}$	0.0842		

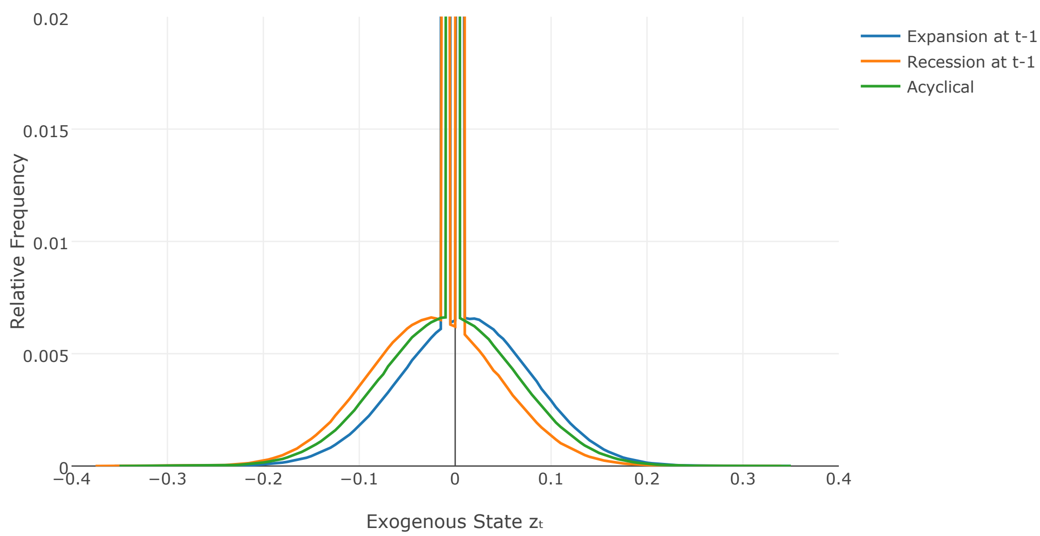
Table 6: Unconditional Moments of y_t^{ss}

	y_t	$\Delta_1 y_t$
Mean	-0.0486	0.0000
Variance	0.0991	0.0154
Skewness	-0.0184	-0.0058
Kurtosis	3.0561	3.0560

To conclude this section, the following figure compares the transition densities $p(z_t|s_{t-1}, z_{t-1} = 0)$ and $p(z_t^{ss}|z_{t-1}^{ss} = 0)$ of the persistent components of the cyclical and acyclical processes, respectively.

[4]Note that the autocovariance function of y_t is $\text{Cov}(y_t, y_{t-k}) = \frac{\rho^k}{1-\rho^2} E[\eta_t(s_t)^2] + \left(\frac{1}{1-\rho} E[\eta_t(s_t)]\right)^2$. Therefore, as $\rho_{ss} = \rho$, $E[\eta_t^{ss}] = E[\eta_t(s_t)]$ and $\text{Var}[\eta_t^{ss}] = \text{Var}[\eta_t(s_t)]$, the autocovariance functions of the cyclical and acyclical processes (y_t and y_t^{ss}) are the same.

Figure 4: Transition Density of Persistent Components z_t and z_t^{ss}



Removing cycles yields a sensible outcome. Targeting long run moments results in a transition density for the acyclical process which is a weighted average of the transition densities conditional on expansion and recession. This in turn has an intuitive meaning. Under the acyclical process there is a constant probability of upward and downward movements (i.e. falling in the right half or in the lower half of the transition density). On the contrary, under the cyclical process the probability of upward moments is higher in expansions and the probability of downward movements is higher in recessions[5].

3 Model

3.1 Environment and Equilibrium

This paper investigates the welfare effects of eliminating business cycles within a model economy similar to that studied by Krusell and Smith (1998). The

[5]Symmetry in the transition densities is a consequence of the symmetry in the transition matrix for the aggregate state s_t. If for instance expansions were more persistent than recessions, than the transition densities would have more mass in the right half.

key differences with respect to their economy concern household's preferences and the idiosyncratic process.

Time is assumed discrete and infinite. In any period, there is a continuum of infinitely-lived households of mass 1. Households have identical recursive preferences of the Epstein and Zin (1989) and Weil (1989) type over consumption streams of the unique good produced and traded in the economy.

Each household is endowed with one unit of time. Her efficiency units $y(s)$, and thus her labor earnings, evolve exogenously according to the process described in the previous section. The distribution of efficiency units depends on the aggregate exogenous state s - either an expansion or a recession. However, conditional on s, individual shocks are independent. Then a law of large numbers holds and the only exogenous source of aggregate risk is the aggregate exogenous state s. I assume that s follows a Markov chain with transition probabilities $\pi_{s,s'}$ between current state s and state s' in the following period.

Heterogeneity across households arises from the assumption of incomplete markets. Idiosyncratic shocks are imperfectly insurable because households can only invest in capital, whose payoff is not contingent on idiosyncratic shocks. Nonetheless, households can use capital to store value and self-insure against idiosyncratic shocks. In the baseline version of the model no unsecured borrowing is allowed, thus individual capital holdings k cannot be negative.

The production side of the economy is the standard one of a stochastic growth model. There is a representative firm producing the unique good with capital input K and labor input L. The production function is assumed Cobb-Douglas with capital share $\alpha \in [0, 1]$: $F(s, K, L) = A_s K^\alpha L^{1-\alpha}$, where A_s is aggregate productivity when the aggregate exogenous state is s.

Capital depreciates at rate $\delta \in [0, 1]$, thus the resource constrain of the closed economy is

$$C + [K' - (1 - \delta)K] = F(s, K, L)$$

Households receive income from labor services and services of their capital. Letting K and L denote aggregate capital and labor supplied in the economy, from the representative firm's problem the wage rate and rental rate of capital are

$$w(K, L, s) = (1-\alpha)A_s \left(\frac{K}{L}\right)^{\alpha} \tag{1}$$

$$r(K, L, s) = \alpha A_s \left(\frac{L}{K}\right)^{(1-\alpha)} - \delta \tag{2}$$

I consider a recursive competitive equilibrium definition. The aggregate state variables of the economy are the current measure μ of households over idiosyncratic states and the aggregate exogenous state s. For a household, the relevant state variables are her current cash-in-hand x, her current realization of the persistent component of efficiency units z, and aggregate states (μ, s) that she will use in order to forecast future prices. A crucial element of the recursive equilibrium with aggregate shocks is the law of motion for the measure μ of households. Given that the only exogenous source of aggregate fluctuations is the aggregate exogenous state s, and given that current idiosyncratic shocks only depend on s (not on s'), the law of motion of μ is described by

$$\mu' = H(\mu, s)$$

for a function H mapping measures into measures.

The optimization problem of the household is

$$\begin{aligned} v(x, z; \mu, s) = \max_{0<c\leq x} \{ &(1-\beta)c^{(1-\theta)} + \\ &+ \beta \mathbb{E}_{z',\epsilon';s'}[v(x', z'; \mu', s')^{\left(\frac{1-\gamma}{1-\theta}\right)} | z; s]^{\left(\frac{1-\theta}{1-\gamma}\right)} \} \end{aligned}$$

subject to the laws of motion

$$x' = w(K', L', s')e^{z'+\epsilon'} + [1 + r(K', L', s')](x - c) \text{ and } \mu' = H(\mu, s)$$

and the exogenous processes for the two components of efficiency units, z and ϵ. Variable c is household's non-durable consumption and $k' \equiv x - c$ is her investment in capital. Function $u(c) = (1-\beta)c^{(1-\theta)}$ is flow-utility from consumption with Epstein-Zin preferences. Parameter β is the quarterly discount factor, $\frac{1}{\theta}$ is the intertemporal elasticity of substitution and γ is the parameter governing risk aversion[6].

[6] Appendix C shows how the expectation on continuation value can be computed, given the structure of the idiosyncratic process.

Therefore a recursive competitive equilibrium is a value function v and policy functions c and k' for the household, firm's choices K and L, price functions r and w and a law of motion H for the measure, such that:

- given prices and the law of motion H, the policy functions solve the household's problem and v is the associated value function;
- given prices, the firm chooses K and L such that equations 1 and 2 hold;
- the capital market clears: $K = \int k'(x, z; \mu, s) d\mu(x, z)$;
- the labor market clears: $L = \int e^{z(s)+\epsilon} d\mu(x, z) dF(\epsilon) = 1$;
- the goods market clears: $\int c(x, z; \mu, s) d\mu(x, z) + K' - (1-\delta)K = F(K, L, s)$;
- the law of motion H is generated by k'.

3.2 Parameter Values and Solution

The following table summarizes all the parameter values set in the model.

Table 7: Model Parameters

Environment		Cyclical Process		Acyclical Process	
β	0.9800	$\pi_{E,E}$	0.8750	ρ_{ss}	0.9935
θ	2/3	$\pi_{R,R}$	0.8750	q_{ss}	0.7641
γ	4.0000	ρ	0.9935	$\sigma_{1,ss}$	0.0004
α	0.3600	q	0.2278	$\sigma_{2,ss}$	0.0708
δ	0.0200	σ_1	0.0674	$\sigma_{\epsilon,ss}$	0.0842
A_E	1.0100	σ_ϵ	0.0842	$\mu_{1,ss}$	0.0006
A_R	0.9900	$\mu_1(E)$	0.0230	$\mu_{2,ss}$	-0.0034
		$\mu_2(E)$	-0.0023		
		$\mu_1(R)$	-0.0266		
		$\mu_2(R)$	0.0075		

Aside from process parameters, the other parameter values are standard in the literature. Kaplan and Violante (2014) also use the same value of the intertemporal substitution parameter θ ($\frac{2}{3}$, implying an intertemporal elasticity of substitution of 1.5) and the risk aversion parameter γ. The value of the discount factor β and capital depreciation rate δ imply a capital-output ratio of 9.23 in the stochastic steady state and an equilibrium return on capital of 1.9%. All the parameters of the Markov chain for the aggregate exogenous state are taken from Krusell and Smith (1998).

Appendix D describes the global solution algorithm used. The law of motion for the measure μ is approximated by a log-linear law of motion for aggregate capital K, following Krusell and Smith (1998).

4 Welfare Effects

4.1 Experiment

In order to investigate the welfare effects of eliminating business cycles in this framework, I follow Storesletten et al. (2001) and consider transitions between four economies:

1. the economy described in the previous section, with aggregate productivity fluctuations and cyclical idiosyncratic process;
2. an economy with cyclical idiosyncratic process only, where aggregate productivity is fixed at its unconditional mean of 1;
3. an economy without business cycles, where prices however are the same as in economy 2;
4. the economy where business cycles have been eliminated.

Considering these four economies allows to decompose the welfare effects of eliminating business cycles - which are the welfare effects of transitioning from economy 1 to economy 4 - into three components: (i) the welfare effects of eliminating fluctuations in aggregate productivity - this is the transition from economy 1 to economy 2; (ii) the welfare effects of eliminating fluctuations in the distribution of the idiosyncratic process - this corresponds to transitioning from economy 2 to 3; (iii) general-equilibrium welfare effects due to differences in prices, which are the residual effects captured through the transition from economy 3 to economy 4.

Welfare effects are defined in consumption-equivalent terms, as the constant proportional change in consumption across any time period and state which makes a household indifferent between living in the current economy or living through the transition to the new economy.

Specifically, let i be the economy where a household with idiosyncratic states (x, z) is currently in, and let j be the economy she can transition to. Let $\omega_{i,j}(x, z; \mu_i, s_i, \mu_{j,1}, s_{j,1})$ be the consumption-equivalent welfare effect for this agent, where $\mu_{j,1}$ denotes the measure on idiosyncratic states in the first period of the transition to economy j and $s_{j,1}$ is the aggregate exogenous state in the same period[7]. Then the welfare effect $\omega_{i,j}$ solves

$$v^i(x, z; \mu_i, s_i; \omega_{i,j}) = v_1^j(x, z; \mu_{j,1}, s_{j,1})$$

where

$$\begin{aligned} v^i(x, z; \mu_i, s_i; \omega_{i,j}) = \max_{0<c\leq x} \{&(1-\beta)(c[1+\omega_{i,j}])^{(1-\theta)} + \\ &+ \beta \mathbb{E}_{z',\epsilon';s_i'}[v^i(x', z'; \mu_i', s_i'; \omega_{i,j})^{\left(\frac{1-\gamma}{1-\theta}\right)}|z; s_i]^{\left(\frac{1-\theta}{1-\gamma}\right)}\} \end{aligned}$$

and v_1^j is value function in the first transition period towards economy j. This value function summarizes the entire value to the household of going through the transition period and then living in economy j. The homogeneity property of the value function yields

$$\omega_{i,j}(x, z; \mu_i, s_i, \mu_{j,1}, s_{j,1}) = \left(\frac{v_1^j(x, z; \mu_{j,1}, s_{j,1})}{v^i(x, z; \mu_i, s_i)}\right)^{\frac{1}{1-\theta}} - 1$$

I consider two ways of summarizing and representing these welfare effects. The first one is the average welfare effect[8]

$$E_{\mu_i}[\omega_{i,j}(x, z; \mu_i, s_i, \mu_{j,1}, s_{j,1})] =$$

[7]When considering a transition to an economy without business cycles, welfare effects are independent of $s_{j,1}$. In such instances, $\omega_{i,j}(x, z; \mu_i, s_i, \mu_{j,1}, s_{j,1}) = \omega_{i,j}(x, z; \mu_i, s_i, \mu_{j,1})$.

[8]Averaging welfare effects over idiosyncratic states could be done either under μ_i or $\mu_{j,1}$. They both result from transitions from the same measure μ_{-1} in the last period before the exogenous, unexpected transition opportunity materializes. Also, both μ_i and $\mu_{j,1}$ are generated by the same household polices. Differences between μ_i and $\mu_{j,1}$ are due to (i) different prices in economy i and in the first transition period towards economy j and (ii) differences in the exogenous processes between economy i and j. The choice of using μ_i is arbitrary, but μ_i and $\mu_{j,1}$ turn out to be fairly similar.

$$= \int\int \omega_{i,j}(x, z; \mu_i, s_i, \mu_{j,1}, s_{j,1}) d\mu_i(x, z) dG(\mu_i, s_i, \mu_{j,1}, s_{j,1})$$

where G is a measure over measures μ_i and $\mu_{j,1}$, and aggregate exogenous states s_i and $s_{j,1}$. Then I consider a representation of welfare effects at the individual level, by averaging only with respect to the aggregate states. The "welfare surface" is then

$$\Omega_{i,j}(x, z) = \int \omega_{i,j}(x, z; \mu_i, s_i, \mu_{j,1}, s_{j,1}) dG(\mu_i, s_i, \mu_{j,1}, s_{j,1})$$

All averages are computed numerically by generating draws from $G(\mu_i, s_i, \mu_{j,1}, s_{j,1})$ through simulation.

4.2 Quantitative Results

The following table reports average welfare effects[9] of eliminating business cycles in three versions of the model. The baseline version is the one described in section 3. In the second version of the model I allow unsecured borrowing by households, so that in the stochastic steady state approximately 8% of households have negative net worth. The third version of the model assumes different preferences for households, leaving everything else unchanged. Households have CRRA preferences over consumption streams, with relative risk aversion equal to 2.

Table 8: Average Welfare Effect of Eliminating Business Cycles

	No cycles	No cycles in A	No id. cycles[a]	Residual
Baseline	8.78%	-0.32%	9.97%	-0.87%
With borrowing[b]	8.81%	-0.32%	10.05%	-0.92%
CRRA = 2	9.25%	-0.24%	10.91%	-1.42%

[a]Same prices as economy without fluctuations in aggregate productivity.
[b]8% of households in steady state have negative net worth.

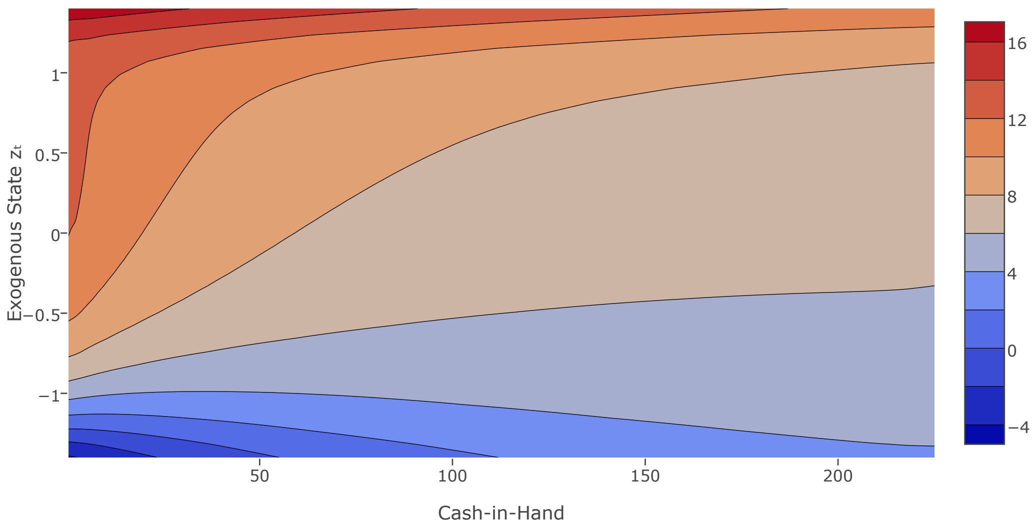
Average welfare effects are similar in magnitude across the different versions of the model. The average welfare effect of eliminating cycles is approximately

[9]Averages are based on 100 draws from $G(\mu_i, s_i, \mu_{j,1}, s_{j,1})$.

9% of lifetime consumption. The third column highlights that all welfare gains come from eliminating fluctuations in the distribution of the idiosyncratic process. The general equilibrium effects are approximately -1% of lifetime consumption. They are caused by differences in aggregate capital between the economy with and without aggregate fluctuations. Aggregate capital is approximately 10% higher in the economy with aggregate fluctuations, therefore the interest rate is lower and the wage is higher compared to the economy where business cycles have been eliminated. Interestingly, the average welfare effect of eliminating only fluctuations in aggregate productivity is slightly negative. I would expect such effect to be closer to zero, as the exogenous process for aggregate productivity is symmetric and the solution method used does not capture precautionary saving motives with respect to aggregate shocks. Increasing the number of simulations used in computing such averages could push the number closer to zero.

Exploring how welfare effects of eliminating business cycles change depending on a household's idiosyncratic states allows to better understand the impact of cyclical skewness of the process. The following contour plot describes the welfare surface $\Omega_{1,4}(x, z)$, i.e. the welfare effects of transitioning from the economy with aggregate fluctuations to the acyclical economy.

Figure 5: Welfare Effects of Eliminating Cycles by Idiosyncratic State (%)



Welfare effects are negative in the bottom-left corner, where cash-in-hand and z (and therefore earnings) at the moment of deciding between the cyclical and acyclical economy are low. Everywhere else welfare effects are positive and attain approximately 16% in the top-left corner.

A household with low initial wealth[10] and earnings[11] prefers the economy with aggregate fluctuations due to an option-value effect. Under both the cyclical and acyclical processes, the household's earnings mean-revert to the same unconditional mean at the same speed, as persistence is the same for the two processes. However, as the household is close to 0 earnings, there is not much room for her earnings to fall more in recessions compared to the economy without business cycles. At the same time, expansions come with a higher probability of upward movements. Moreover, wealth is also close to its lower bound and cannot fall much further during recessions in the cyclical economy. For these reasons, low-wealth, low-productivity households prefer the economy

[10]Cash-in-hand and capital holdings, although different, are similar in value and distribution in the model.

[11]Earnings $we^{z+\epsilon}$ are heavily dependent on the persistent component z of efficiency units/productivity.

with aggregate fluctuations.

The option-value effect becomes less important as wealth and productivity increase. Given a low value of z, welfare effects increase and turn positive with higher cash-in-hand as there is now room for a differential impact of the cyclical process in bad times on household's wealth and eventually consumption. Given a low value of cash-in-hand, welfare effects are increasing in z as the higher probability of downward movements in earnings during recessions starts to bite.

Finally, for middle to high values of productivity, welfare effects are decreasing in wealth, while remaining positive. This trend is due to larger self-insurance capacity afforded by higher initial wealth, so that cyclicality in the distribution of productivity becomes relatively less important for household's consumption when initial wealth increases.

5 Conclusions

This paper has investigated the quantitative implications of the extent of countercyclical labor earnings risk measured by Guvenen et al. (2014). Using a standard heterogeneous agent, general equilibrium model with aggregate shocks and state-dependent idiosyncratic shocks I find that the average welfare effect of transitioning to an economy without business cycles is approximately 9% of lifetime consumption. I also highlight that the welfare gain comes entirely from eliminating cyclical changes in idiosyncratic risk. Results are robust to different preference specifications (Epstein-Zin or CRRA) and allowing for unsecured borrowing.

Looking at welfare effects at the household level, over most of the idiosyncratic state space welfare gains are increasing in individual, persistent productivity and decreasing in wealth. However, low-productivity, low-wealth households prefer the economy with aggregate fluctuations due to an option-value effect. Being close to the lower bound, the higher probability of falling to lower earnings in recessions does not have much bite. On the contrary, they do enjoy the higher probability of upward movements during expansions, compared to the economy without business cycles.

Having established that this extent of countercyclical risk can have quantitatively relevant implications, the next step is introducing an endogenous mechanism which matches cyclical changes in the labor earnings distribution and allows for structural policy analysis. The "ladder" mechanisms developed by Lise (2013) and Jarosch (2015) are promising in this respect. I leave this as an avenue for future work.

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Appendix A

Let $\tilde{y}_t$ be the quarterly earnings process conditional on $s_t = s$ for all t. Recall that the process specification is

$$\tilde{y}_t = \tilde{z}_t + \tilde{\epsilon}_t$$

$$\tilde{z}_t = \rho\tilde{z}_{t-1} + \tilde{\eta}_t$$

$$\tilde{\epsilon}_t \sim \mathrm{N}(0, \sigma_\epsilon)$$

$$\tilde{\eta}_t = \tilde{\xi}_t\tilde{\eta}_{1,t} + (1 - \tilde{\xi}_t)\tilde{\eta}_{2,t}$$

$$\tilde{\eta}_{j,t} \sim \mathrm{N}(\mu_j(s), \sigma_j) \text{ for } j = 1, 2$$

$$\tilde{\xi}_t \sim \text{Bernoulli}(q)$$

Let

$$\mathcal{Y}_t = \sum_{j=0}^{3} \tilde{y}_{t-j}$$

Then

$$\mathcal{Y}_t = \sum_{j=0}^{3} \tilde{z}_{t-j} + \sum_{j=0}^{3} \tilde{\epsilon}_{t-j}$$

As $\tilde{\epsilon}_t$ are *i.i.d.*,

$$\mathcal{E}_t = \sum_{j=0}^{3} \tilde{\epsilon}_{t-j} \sim \mathrm{N}(0, 4\sigma_\epsilon)$$

Substituting recursively I also get

$$\begin{aligned}
\mathcal{Z}_t &= \sum_{j=0}^{3} \tilde{z}_{t-j} \\
&= \tilde{\eta}_t + (1+\rho)\tilde{\eta}_{t-1} + (1+\rho+\rho^2)\tilde{\eta}_{t-2} + (1+\rho+\rho^2+\rho^3)\tilde{z}_{t-3} \\
&= \sum_{j=0}^{2} [G(j)\tilde{\eta}_{t-j}] + G(3)\tilde{z}_{t-3}
\end{aligned}$$

where

$$G(j) = \sum_{i=0}^{j} \rho^i$$

Let $S[X]$ and $K[X]$ denote skewness and kurtosis of a random variable X. Then the moments of the level of the annualized quarterly process $\mathcal{Y}_t$ are

$$\begin{aligned} E[\mathcal{Y}_t] &= E[\mathcal{Z}_t] + E[\mathcal{E}_t] \\ &= \left(\sum_{j=0}^{2} G(j)\right) E[\tilde{\eta}_t] + G(3)E[\tilde{z}_t] \end{aligned} \tag{1}$$

$$\begin{aligned} \text{Var}[\mathcal{Y}_t] &= \text{Var}[\mathcal{Z}_t] + \text{Var}[\mathcal{E}_t] + 2\text{Cov}[\mathcal{Z}_t, \mathcal{E}_t] \\ &= \left(\sum_{j=0}^{2} G(j)^2\right) \text{Var}[\tilde{\eta}_t] + G(3)^2\text{Var}[\tilde{z}_t] + 4\sigma_\epsilon^2 + 0 \end{aligned} \tag{2}$$

$$S[\mathcal{Y}_t] = \left(\sum_{j=0}^{2} G(j)^3\right) \left(\frac{\text{Var}[\tilde{\eta}_t]}{\text{Var}[\mathcal{Y}_t]}\right)^{\frac{3}{2}} S[\tilde{\eta}_t] + G(3)^3\text{Var}\left(\frac{\text{Var}[\tilde{z}_t]}{\text{Var}[\mathcal{Y}_t]}\right)^{\frac{3}{2}} S[\tilde{z}_t] \tag{3}$$

$$K[\mathcal{Y}_t] = 3+\left(\sum_{j=0}^{2} G(j)^4\right) \left(\frac{\text{Var}[\tilde{\eta}_t]}{\text{Var}[\mathcal{Y}_t]}\right)^{2} (K[\tilde{\eta}_t] - 3)+G(3)^4\text{Var}\left(\frac{\text{Var}[\tilde{z}_t]}{\text{Var}[\mathcal{Y}_t]}\right)^{2} (K[\tilde{z}_t] - 3) \tag{4}$$

Let $C_k[X]$ denote the k-th cumulant of the random variable X. Note that (3) and (4) use the following implications of properties of cumulants of a random variable:

$$\begin{aligned} C_k[\mathcal{Y}_t] &= C_k[\mathcal{Z}_t] + C_k[\mathcal{E}_t] \\ &= \left(\sum_{j=0}^{2} G(j)^k\right) C_k[\tilde{\eta}_t] + G(3)^k C_k[\tilde{z}_t] + C_k[\mathcal{E}_t] \end{aligned}$$

$$C_3[X] = \text{Var}[X]^{\frac{3}{2}} S[X]$$

$$C_4[X] = \text{Var}[X]^2 (K[X] - 3)$$

Finally, the moments of $\tilde{z}_t$ and $\tilde{\eta}_t$ can be computed using the formulas developed by Civale et al. (2015).

Expressions for the moments of the h-year difference of the annualized quarterly process $\mathcal{Y}_t$ are as follows. I have

$$\begin{aligned} \Delta_h \mathcal{Y}_t &= \Delta_h \mathcal{Z}_t + \Delta_h \mathcal{E}_t \\ &= (\mathcal{Z}_t - \mathcal{Z}_{t-4h}) + (\mathcal{E}_t - \mathcal{E}_{t-4h}) \end{aligned}$$

Focusing on the first term, I get

$$
\begin{aligned}
\mathcal{Z}_t - \mathcal{Z}_{t-4h} &= \sum_{j=0}^{2} \left[G(j)\tilde{\eta}_{t-j} \right] + G(3)\tilde{z}_{t-3} - \sum_{j=0}^{2} \left[G(j)\tilde{\eta}_{t-4h-j} \right] - G(3)\tilde{z}_{t-3-4h} \\
&= \sum_{j=0}^{2} \left[G(j)\tilde{\eta}_{t-j} \right] + G(3) \sum_{k=0}^{\infty} \rho^k \tilde{\eta}_{t-3-k} + \\
&\quad - \sum_{j=0}^{2} \left[G(j)\tilde{\eta}_{t-4h-j} \right] - G(3) \sum_{k=0}^{\infty} \rho^k \tilde{\eta}_{t-3-4h-k}
\end{aligned} \tag{5}
$$

The term

$$G(3) \sum_{k=0}^{\infty} \rho^k \tilde{\eta}_{t-3-k}$$

can be split up as

$$G(3) \sum_{k=0}^{\infty} \rho^k \tilde{\eta}_{t-3-k} = G(3) \sum_{k=0}^{4(h-1)} \rho^k \tilde{\eta}_{t-3-k} + G(3)\rho^{4h-3} \sum_{k=0}^{\infty} \rho^k \tilde{\eta}_{t-4h-k}$$

so (5) becomes

$$
\begin{aligned}
\mathcal{Z}_t - \mathcal{Z}_{t-4h} &= \sum_{j=0}^{2} \left[G(j)\tilde{\eta}_{t-j} \right] + G(3) \sum_{k=0}^{4(h-1)} \rho^k \tilde{\eta}_{t-3-k} + \\
&\quad + \sum_{j=0}^{2} \left[\left\{ G(3)\rho^{4h-3+j} - G(j) \right\} \tilde{\eta}_{t-4h-j} \right] + \\
&\quad + G(3) \left(\rho^{4h-3} \sum_{k=3}^{\infty} \rho^k \tilde{\eta}_{t-4h-k} - \sum_{k=0}^{\infty} \rho^k \tilde{\eta}_{t-3-4h-k} \right)
\end{aligned} \tag{6}
$$

Now I focus on the term in parenthesis in equation (6). That term can be rewritten as

$$
\begin{aligned}
\rho^{4h-3} \sum_{k=3}^{\infty} \rho^k \tilde{\eta}_{t-4h-k} - \sum_{k=0}^{\infty} \rho^k \tilde{\eta}_{t-3-4h-k} &= \rho^{4h-3}\rho^3 \sum_{k=0}^{\infty} \rho^k \tilde{\eta}_{t-3-4h-k} - \sum_{k=0}^{\infty} \rho^k \tilde{\eta}_{t-3-4h-k} \\
&= (\rho^{4h} - 1) \sum_{k=0}^{\infty} \rho^k \tilde{\eta}_{t-3-4h-k}
\end{aligned}
$$

and (6) finally becomes

$$\begin{aligned}\mathcal{Z}_t - \mathcal{Z}_{t-4h} = &\sum_{j=0}^{2}\left[G(j)\tilde{\eta}_{t-j}\right] + G(3)\sum_{k=0}^{4(h-1)}\rho^k\tilde{\eta}_{t-3-k} + \\ &+ \sum_{j=0}^{2}\left[\left\{G(3)\rho^{4h-3+j} - G(j)\right\}\tilde{\eta}_{t-4h-j}\right] + \\ &+ G(3)(\rho^{4h}-1)\sum_{k=0}^{\infty}\rho^k\tilde{\eta}_{t-3-4h-k}\end{aligned} \tag{7}$$

yielding

$$\begin{aligned}\Delta_h\mathcal{Y}_t = &\sum_{j=0}^{2}\left[G(j)\tilde{\eta}_{t-j}\right] + G(3)\sum_{k=0}^{4(h-1)}\rho^k\tilde{\eta}_{t-3-k} + \\ &+ \sum_{j=0}^{2}\left[\left\{G(3)\rho^{4h-3+j} - G(j)\right\}\tilde{\eta}_{t-4h-j}\right] + \\ &+ G(3)(\rho^{4h}-1)\sum_{k=0}^{\infty}\rho^k\tilde{\eta}_{t-3-4h-k} + (\mathcal{E}_t - \mathcal{E}_{t-4h})\end{aligned}$$

Using properties of cumulants, I obtain that the n-th cumulant of $\Delta_h\mathcal{Y}_t$ is

$$\begin{aligned}C_n[\Delta_h\mathcal{Y}_t] = &C_n[\tilde{\eta}_t]\sum_{j=0}^{2}G(j)^n + C_n[\tilde{\eta}_t]G(3)^n\sum_{k=0}^{4(h-1)}\rho^{kn} + \\ &+ C_n[\tilde{\eta}_t]\left[\sum_{j=0}^{2}\left\{G(3)\rho^{4h-3+j} - G(j)\right\}^n\right] + \\ &+ G(3)^n(\rho^{4h}-1)^n\frac{C_n[\tilde{\eta}_t]}{1-\rho^n} + C_n[\mathcal{E}_t] + C_n[-\mathcal{E}_t]\end{aligned}$$

which can be written compactly using

$$\begin{aligned}\Gamma(h,n) = = &\sum_{j=0}^{2}G(j)^n + G(3)^n\frac{1-\rho^{n(4h-3)}}{1-\rho^n} + \\ &+ \left[\sum_{j=0}^{2}\left\{G(3)\rho^{4h-3+j} - G(j)\right\}^n\right] + \\ &+ G(3)^n(\rho^{4h}-1)^n\frac{1}{1-\rho^n}\end{aligned}$$

as

$$C_n[\Delta_h \mathcal{Y}_t] = C_n[\tilde{\eta}_t]\Gamma(h,n) + C_n[\mathcal{E}_t] + C_n[-\mathcal{E}_t]$$

Then, again using properties of cumulants, the moments of the h-year difference of the annualized quarterly process $\mathcal{Y}_t$ are

$$E[\Delta_h \mathcal{Y}_t] = 0$$

$$\mathrm{Var}[\Delta_h \mathcal{Y}_t] = \mathrm{Var}[\tilde{\eta}_t]\Gamma(h,2) + 8\sigma_\epsilon^2$$

$$S[\Delta_h \mathcal{Y}_t] = S[\tilde{\eta}_t]\Gamma(h,3)\left(\frac{\mathrm{Var}[\tilde{\eta}_t]}{\mathrm{Var}[\Delta_h \mathcal{Y}_t]}\right)^{\frac{3}{2}}$$

$$K[\Delta_h \mathcal{Y}_t] = 3 + (K[\tilde{\eta}_t] - 3)\Gamma(h,4)\left(\frac{\mathrm{Var}[\tilde{\eta}_t]}{\mathrm{Var}[\Delta_h \mathcal{Y}_t]}\right)^{2}$$

where the moments of $\tilde{\eta}_t$ can be computed using the formulas developed by Civale et al. (2015).

The following table reports ratios between target yearly moments and moments of the annualized quarterly process in order to provide information on the accuracy of the parameterization of the quarterly process. As there are more moments than parameters, distance minimization is used. All moments are weighted equally in the objective function.

Table 1: Comparison of Moments between Yearly (y^a) and Annualized Quarterly Process ($\mathcal{Y}$)

		Expansion	
	$\frac{\text{Moment}_{\mathcal{Y}}}{\text{Moment}_{y^a}}$	$\frac{\text{Moment}_{\Delta_1 \mathcal{Y}}}{\text{Moment}_{\Delta_1 y^a}}$	$\frac{\text{Moment}_{\Delta_5 \mathcal{Y}}}{\text{Moment}_{\Delta_5 y^a}}$
Mean	1.0000	-	-
Variance	1.0211	0.8401	1.1006
Skewness	1.0519	0.9186	1.0202
Kurtosis	1.0023	0.9873	1.0036

		Recession	
	$\frac{\text{Moment}_{\mathcal{Y}}}{\text{Moment}_{y^a}}$	$\frac{\text{Moment}_{\Delta_1 \mathcal{Y}}}{\text{Moment}_{\Delta_1 y^a}}$	$\frac{\text{Moment}_{\Delta_5 \mathcal{Y}}}{\text{Moment}_{\Delta_5 y^a}}$
Mean	1.0000	-	-
Variance	1.0207	0.8407	1.1040
Skewness	1.0537	0.9188	1.0168
Kurtosis	1.0038	1.0021	1.0120

The following plots offer another representation of the accuracy of the quarterly parameterization by comparing the conditional stationary densities of the transformations of the yearly process and the annualized quarterly process.

Figure 1: Yearly vs. Annualized Quarterly Process, Log-Earnings

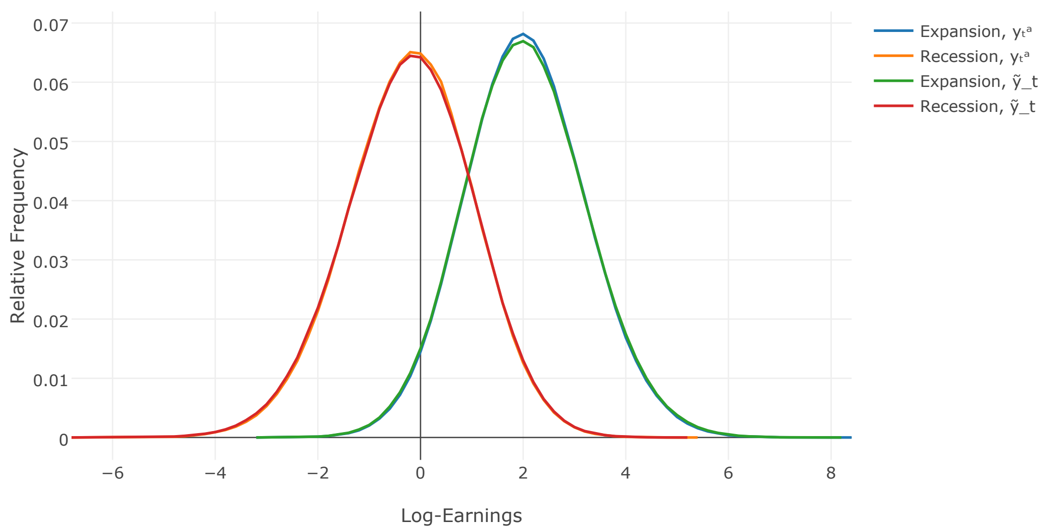


Figure 2: Yearly vs. Annualized Quarterly Process, First Difference

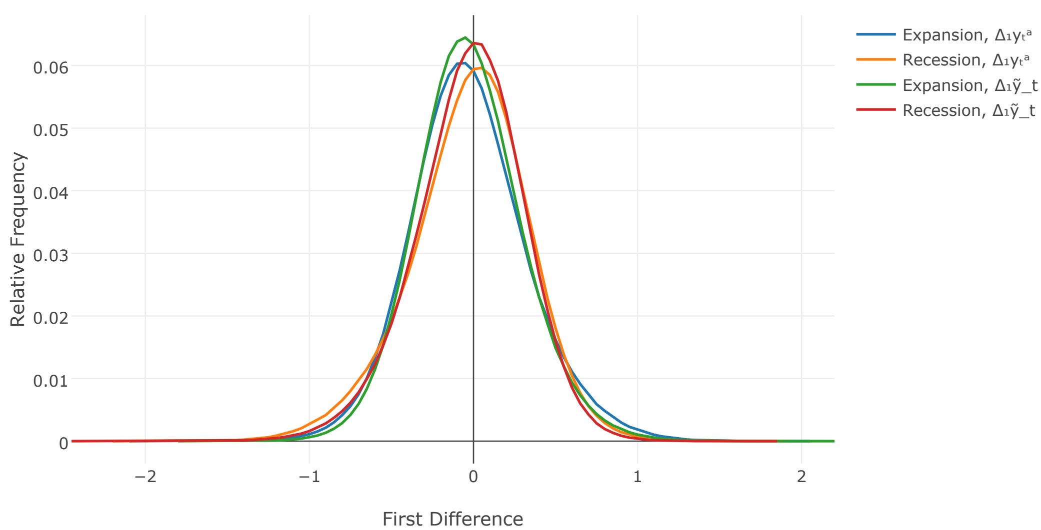
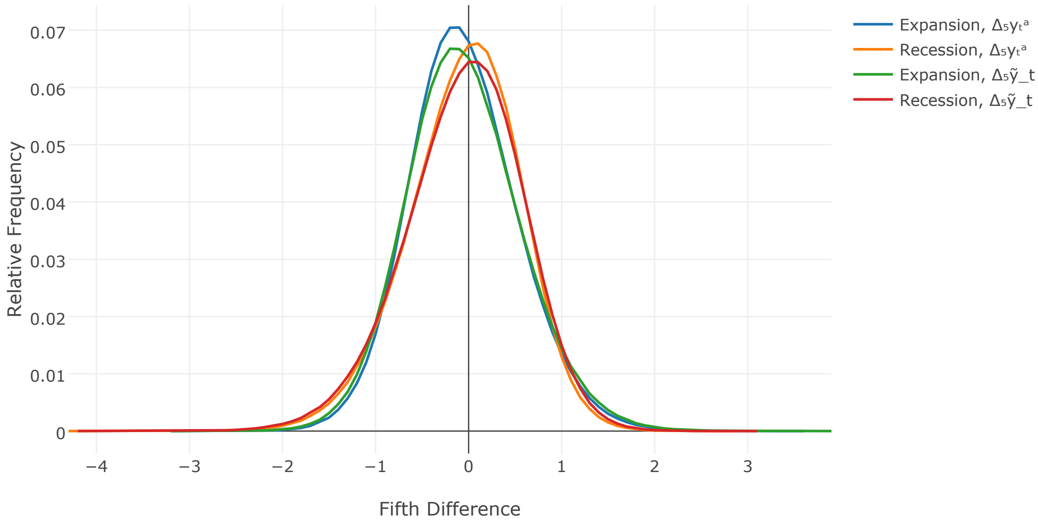


Figure 3: Yearly vs. Annualized Quarterly Process, Fifth Difference



Appendix B

The following plots compare the stationary densities of the cyclical process and the acyclical process. Despite targeting only the first four moments, the densities align very well. This is true also for the density of the fifth difference transformation, whose moments are not explicitly targeted.

Figure 1: Approximation of Stationary Density, Log-Efficiency Units

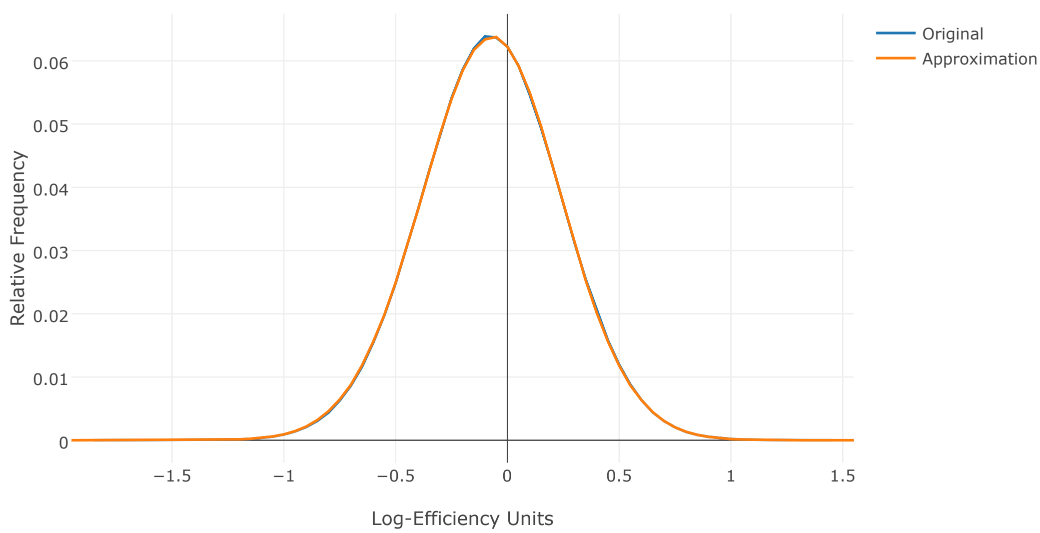


Figure 2: Approximation of Stationary Density, First Difference

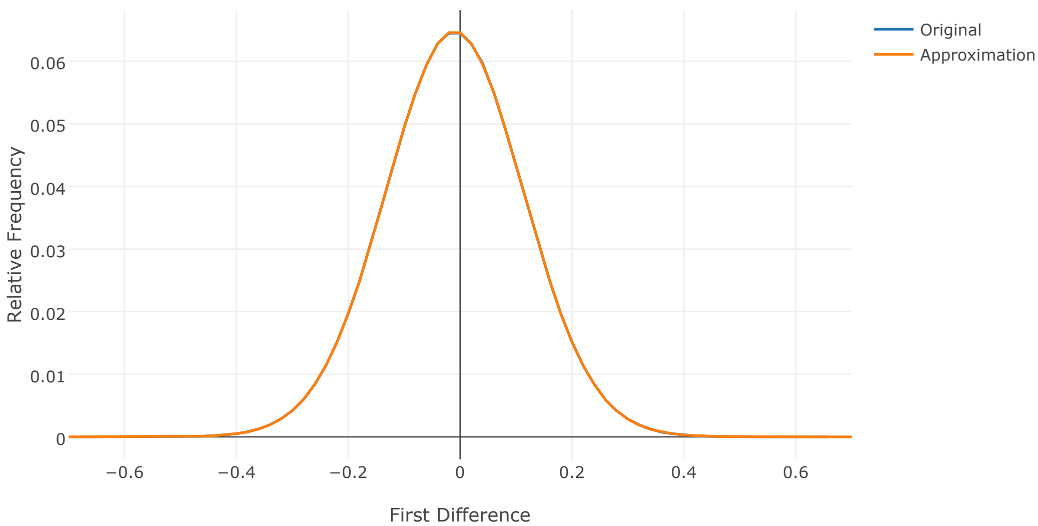
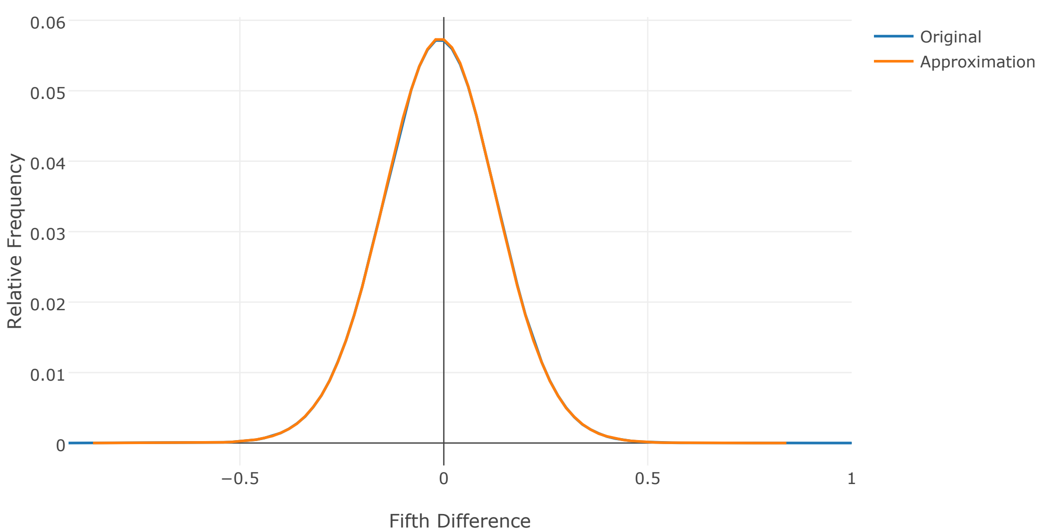


Figure 3: Approximation of Stationary Density, Fifth Difference



Appendix C

Using independence among ϵ', ξ' and η' conditional on s', it is possible to write

$$\begin{aligned}
\mathbb{E}_{z',\epsilon';s'}[v(x',z';\mu',s')^{\left(\frac{1-\gamma}{1-\theta}\right)}|z;s] &= \mathbb{E}_{s'}\left[\mathbb{E}_{z',\epsilon'}[v(x',z';\mu',s')^{\left(\frac{1-\gamma}{1-\theta}\right)}|s']|s,z\right] \\
&= \pi_{s,E}\mathbb{E}_{z',\epsilon'}[v_E(x',z';\mu')^{\left(\frac{1-\gamma}{1-\theta}\right)}|z] + \pi_{s,R}\mathbb{E}_{z',\epsilon'}[v_R(x',z';\mu')^{\left(\frac{1-\gamma}{1-\theta}\right)}|z] \\
&= \pi_{s,E}\{q\mathbb{E}_{\eta'_{1,E},\epsilon'}[v_E(x',-(1-\rho)C_{\tilde{y}|E}(1)+\rho z+\eta'_{1,E};\mu')^{\left(\frac{1-\gamma}{1-\theta}\right)}|z]+ \\
&+ (1-q)\mathbb{E}_{\epsilon'}[v_E(x',-(1-\rho)C_{\tilde{y}|E}(1)+\rho z+\mu_{2,E};\mu')^{\left(\frac{1-\gamma}{1-\theta}\right)}|z]\}+ \\
&+ \pi_{s,R}\{q\mathbb{E}_{\eta'_{1,R},\epsilon'}[v_R(x',-(1-\rho)C_{\tilde{y}|R}(1)+\rho z+\eta'_{1,R};\mu')^{\left(\frac{1-\gamma}{1-\theta}\right)}|z]+ \\
&+ (1-q)\mathbb{E}_{\epsilon'}[v_R(x',-(1-\rho)C_{\tilde{y}|R}(1)+\rho z+\mu_{2,R};\mu')^{\left(\frac{1-\gamma}{1-\theta}\right)}|z]\}
\end{aligned}$$

Appendix D

The approximate solution to the model is computed by approximating the value functions using linear B-splines with collocation method. With the approximate value functions at hand I can compute the approximate policy functions as linear interpolants. I use tolerance of 10^{-10} in the approximation to the value functions and I solve the asset-allocation problem using a golden-search algorithm[1]. I approximate the law of motion of the measure μ on idiosyncratic states by a log-linear function of current exogenous aggregate state s and current aggregate capital K, as in Krusell and Smith (1998). Then the approximate solution to the model is computed using fixed point iteration on the law of motion of aggregate capital, updating the law of motion with least-squares projection. This requires simulating a history of aggregate states s and μ, which I do by stochastic simulation[2] of a population of 20,000 households for 4,000 periods, with a 1,000 periods burn-in in order to eliminate the effect of initial conditions. Iterations on the law of motion are stopped at tolerance 10^{-4}.

[1] I rely on the translation into Julia of the Miranda and Fackler CompEcon routines. The Julia CompEcon routines have been written mostly by the NYU Stern PhD student Spencer Lyon with some contribution from myself.

[2] I find that stochastic simulation - compared to non-stochastic simulation *a la* Young (2010) - gives a more accurate representation of the state-dependent distribution of households over efficiency units.

Regarding transition paths when eliminating business cycles, I use the solution to the model with aggregate fluctuations to generate 100 draws from the stationary distribution of the model. For each of these draws, I use a shooting algorithm to solve for the transition path. I guess a time horizon when the transition is completed (60 periods in case of the transition to the acyclical economy) and a path for aggregate capital along the transition. Then I solve the household problem in each transition period by backward induction. Having the policy functions in each period, I can simulate the measure μ of households along the transition. This implies an endogenous path for aggregate capital which I use to update the initial guess. The fixed point iteration on the transition path of aggregate capital is stopped at tolerance 10^{-3}.

As concerns the grids on cash-in-hand x and the persistent component of efficiency units z, I set the grid limits for z to [-1.4,1.4], which includes both the 0.01th percentile of the simulated limiting distribution of z conditional on recession and the 99.99th percentile of its simulated limiting distribution conditional on expansion. I use 75 grid points for interpolating x on [0.32,230.27] and 35 grid points for z [3]. In order to compute expectations with respect to the exogenous variables ϵ, η_1 and η_2 I use Gaussian quadrature with Hermite polynomials. This is the approach followed by McKay (2016), for instance. Specifically, I use 5 points each for $\eta_1(E)$, $\eta_1(R)$, $\eta_{1,ss}$, $\eta_{2,ss}$ and ϵ. Grid points for x are spaced according to the formula $x_i = \underline{x} + \left[\frac{i}{75}(\bar{x} - \underline{x})^{0.4}\right]^{\frac{1}{0.4}}$ in order to concentrate more of them on low values of x, where the value functions have more curvature. Grid points for z are evenly spaced. Finally, I use a 5-point grid for aggregate capital K on [30.5, 36.5] and verify that aggregate capital always remains within the grid when the approximate solution is simulated.

The following table summarizes accuracy measures for the approximate solution. All measures are computed based on a different history of the exogenous aggregate state than the one used when solving the fixed point problem. Values are in line, if not better, than those reported by McKay (2016).

[3] The bounds on the x space are determined by taking the widest possible interval for cash in hand given prices, the quadrature points for η and ϵ and assets bounds $\underline{a} = 0$, $\bar{a} = 225$.

Approximation Measure	Value
Max Absolute Difference	0.29%
Mean Absolute Difference	0.09%
R^2, Expansion	0.9998
R^2, Recession	0.9997
RMSE, Expansion	0.0065
RMSE, Recession	0.0067

As a visual check, I also plot the path of log capital for a given history of the aggregate state using only the law of motion vs. simulating the history of the measure μ.

Figure 1: Log Capital Stock, Simulated vs. Implied Values from Forecast Rule